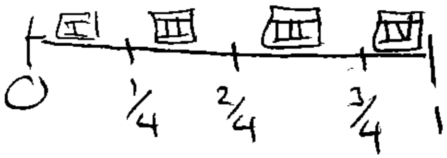
Entry Task (*you do*): Approx. the area under $f(x) = x^3$ from x = 0 to x = 1 using n = 4 and *right-endpoints*.

Step 1: $\Delta x = \frac{b-a}{n} = \frac{1-0}{4} = 1/4$

Step 2:

$$
\begin{aligned}
x_0 &= a = 0 \\
x_1 &= a + \Delta x = 0 + 1/4 \\
x_2 &= a + 2\Delta x = 0 + 2(1/4) \\
x_3 &= a + 3\Delta x = 0 + 3(1/4) \\
x_4 &= a + 4\Delta x = 0 + 4(1/4)
\end{aligned}
$$



Step 3: Plug in right-endpoints to function to get rect. heights, then add up areas (height times width).

$$Area \approx \sum_{i=1}^{4} f(x_i)\Delta x =$$

$$\underbrace{f(x_1)}\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + f(x_4)\Delta x$$

$$(x_1)^3 \frac{1}{4} + (x_2)^3 \frac{1}{4} + (x_3)^3 \frac{1}{4} + (x_4)^3 \frac{1}{4}$$

$$\underset{i=1}{\left(\frac{1}{4}\right)^3 \frac{1}{4}} + \underset{i=2}{\left(\frac{2}{4}\right)^3 \frac{1}{4}} + \underset{i=3}{\left(\frac{3}{4}\right)^3 \frac{1}{4}} + \underset{i=4}{\left(\frac{4}{4}\right)^3 \frac{1}{4}}$$

PATTERN: $\left(\frac{i}{4}\right)^3 \frac{1}{4}$

SAME

SHORTHAND:

$$\sum_{i=1}^{4} \left(\frac{i}{4}\right)^3 \frac{1}{4}$$

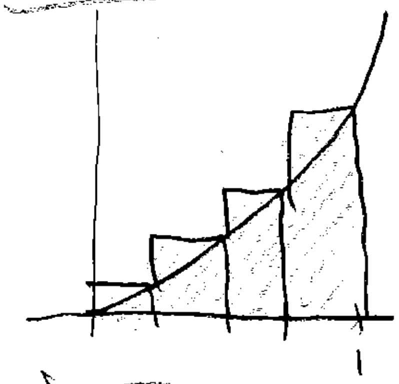
$$\approx 0.390625$$

I did this example again with 100 subdivisions, then 1000, then 10000. Here is a summary of my findings:

n	R_n	L_n
4	0.390625	0.140625
5	0.36	0.16
10	0.3025	0.2025
100	0.255025	0.245025
1000	0.25050025	0.24950025
10000	0.2499500025	0.2500500025

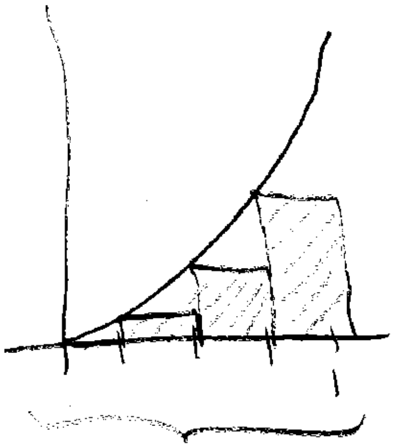
oops

VISUAL



R_4

OVERESTIMATE



L_4

UNDERESTIMATE

Pattern:

$$\Delta x = \frac{1-0}{n} = \frac{1}{n}, \quad x_i = 0 + i\frac{1}{n} = \frac{i}{n}$$

$$\text{Sum} = \sum_{i=1}^{n} x_i^3 \Delta x = \sum_{i=1}^{n} \left(\frac{i}{n}\right)^3 \frac{1}{n}$$

$$\text{Area} = 0.25 = \underbrace{\lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{i}{n}\right)^3 \frac{1}{n}}_{\text{WE CALL THIS THE EXACT AREA AND DENOTE IT}}$$

$$\boxed{\begin{aligned} \int_0^1 x^3 dx &= \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{i}{n}\right)^3 \frac{1}{n} \\ &= \lim_{n\to\infty} \sum_{i=1}^{n} x_i^3 \Delta x \end{aligned}}$$

Example: Approximate the area under $f(x) = 1 + x^2$ from $x = 2$ to $x = 3$ using Riemann sums with $n = 4$ and right endpoints.

$$\Delta x = \frac{3-2}{4} = 1/4$$

$$x_0 = 2$$
$$x_1 = 2 + 1/4 = 2.25$$
$$x_2 = 2 + 2(1/4) = 2.5$$
$$x_3 = 2 + 3(1/4) = 2.75$$
$$x_4 = 2 + 4(1/4) = 3 \checkmark$$

$$\underset{\boxed{I}}{(1+2.25^2)\tfrac{1}{4}} + \underset{\boxed{II}}{(1+2.5^2)\tfrac{1}{4}} + \underset{\boxed{III}}{(1+2.75^2)\tfrac{1}{4}} + \underset{\boxed{IV}}{(1+3^2)\tfrac{1}{4}}$$

$$= 7.96875$$

What is the general pattern in terms of *n*?

$$\Delta x = \frac{3-2}{n} = \frac{1}{n}$$

$$x_i = a + i\Delta x = 2 + \frac{i}{n}$$

$$\sum_{i=1}^{n} f(x_i)\Delta x = \sum_{i=1}^{n} (1 + x_i^2)\Delta x$$
$$= \sum_{i=1}^{n} \left(1 + \left(2 + \frac{i}{n}\right)^2\right)\frac{1}{n}$$

$$\boxed{\begin{aligned} &\int_2^3 1 + x^2 \, dx \\ &= \lim_{n\to\infty} \sum_{i=1}^{n} (1 + x_i^2)\Delta x \\ &= \lim_{n\to\infty} \sum_{i=1}^{n} \left(1 + (\underset{\underset{a}{\uparrow}}{2} + \tfrac{i}{n})^2\right) \frac{1}{n} \end{aligned}} \leftarrow b-a$$

Another Example:

Using sigma notation, write down the general Riemann sum definition of the area from x = 5 to x = 7 under

$$f(x) = 3x + \sqrt{x}$$

$$\Delta x = \frac{b-a}{n} = \frac{7-5}{n} = \frac{2}{n}$$

$$x_i = a + i\,\Delta x = 5 + i\left(\frac{2}{n}\right) = 5 + \frac{2i}{n}$$

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x = \boxed{\begin{aligned} &\lim_{n\to\infty} \sum_{i=1}^{n} \left(3x_i + \sqrt{x_i}\right)\Delta x \\ &= \lim_{n\to\infty} \sum_{i=1}^{n} \left(3\left(5+\frac{2i}{n}\right) + \sqrt{5+\frac{2i}{n}}\right)\frac{2}{n} = \int_5^7 3x + \sqrt{x}\,dx \end{aligned}}$$

$b - a = 2$

$x_i = \underset{\substack{\uparrow \\ a}}{5} + i\left(\frac{2}{n}\right)$

5.2 The Definite Integral

Def'n: We define the **definite integral of f(x) from x = a to x = b** by

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x,$$

where $\Delta x = \frac{b-a}{n}$ and $x_i = a + i\Delta x$.

NOTES

"$\int$" IS CALLED THE INTEGRAL SIGN

a, b ARE THE BOUNDS (OR LIMITS) OF INTEGRATION.

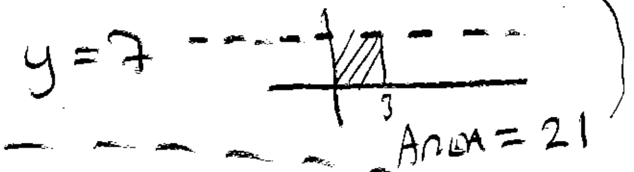
$$\int_a^b f(x)dx = \underline{\text{A NUMBER}}$$

$$= \begin{cases} \text{THE SUM OF ADDING UP} \\ f(x_i)\Delta x \text{ WITH} \\ \text{SMALLER AND SMALLER} \\ \text{SUBDIVISIONS} \end{cases}$$

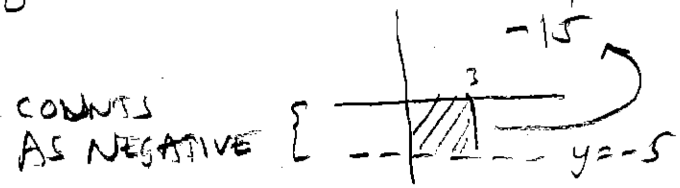
$$\int_a^b f(x)dx = \begin{cases} \text{THE "NET"} \\ \text{(OR "SIGNED")} \\ \text{AREA BETWEEN} \\ f(x) \text{ AND THE} \\ \text{X-AXIS} \end{cases}$$

Ex)

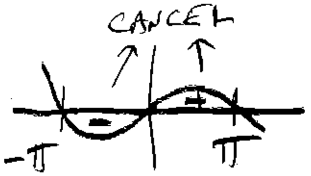
$$\int_0^3 7dx = 21$$



$$\int_0^3 -5\,dx = -15$$



$$\int_{-\pi}^{\pi} \sin(x)dx = 0$$



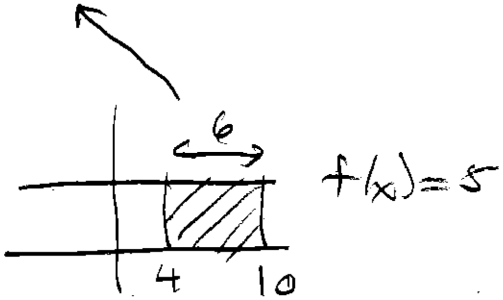
Basic Integral Rules:

1. $\int_a^b c\,dx = (b-a)c$

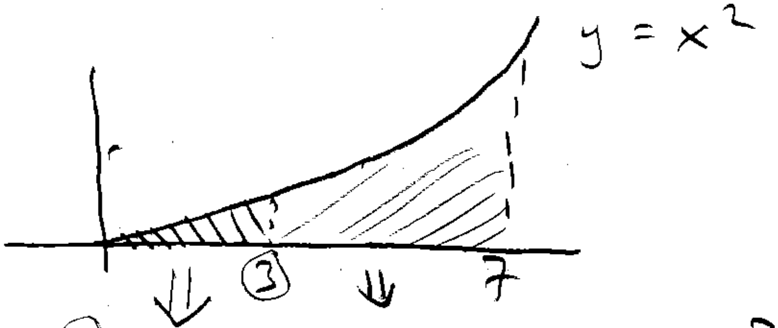
2. $\int_a^b f(x)dx + \int_b^c f(x)dx = \int_a^c f(x)dx$

Examples:

1. $\int_4^{10} 5\,dx = (10-4)\cdot 5 = 30$



2. $\int_0^3 x^2 dx + \int_3^7 x^2 dx =$



$$\int_0^{\textcircled{3}} x^2 dx + \int_{\textcircled{3}}^7 x^2 dx = \int_0^7 x^2 dx$$

Basic Integral Rules:

3. $\int_a^b cf(x)\,dx = c\int_a^b f(x)dx$

and

$$\int_a^b f(x) + g(x)\,dx$$

$$= \int_a^b f(x)dx + \int_a^b g(x)\,dx$$

Examples:

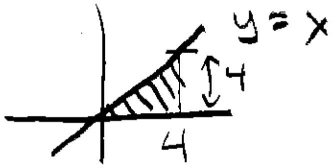
3. $\int_0^4 5x + 3\,dx =$

$$= \int_0^4 5x\,dx + \int_0^4 3\,dx$$

$$= 5\underbrace{\int_0^4 x\,dx} + 3\int_0^4 1\,dx$$

$$= \underbrace{5 \cdot \tfrac{1}{2}(4)(4)} + 3 \cdot 4 = 40 + 12$$

$$= 52$$



4. $\int_b^a f(x)dx = -\int_a^b f(x)dx$

4. $\int_3^1 x^3 dx = -\int_1^3 x^3\,dx$

NOTE: IN THE CONSTRUCTION THE ONLY DIFFERENCE IS $a = 3$ AND $b = 1$ SO $\Delta x = \frac{b-a}{n} = \frac{1-3}{n} = \frac{-2}{n}$

INSTEAD OF $a = 1$ AND $b = 3$ SO $\Delta x = \frac{b-a}{n} = \frac{3-1}{n} = \frac{2}{n}$

OPPOSITE SIGN

Note on quick bounds (in HW_1C)

$$m(b-a) \le \int_a^b f(x)dx \le M(b-a)$$

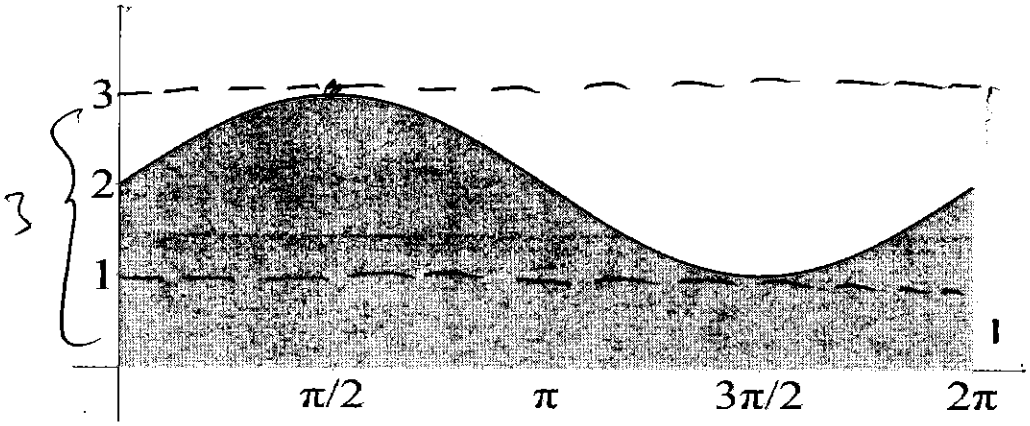
Example: Consider the area under

f(x) = sin(x) + 2

from *x = 0* to *x = 2π.*

(a) What is the max of f(x)? (label M)
(b) What is the min of f(x)? (label m)

(c) Draw **one** rectangle with width *2π* and height M?
(d) Draw **one** rectangle with width *2π* and height m?

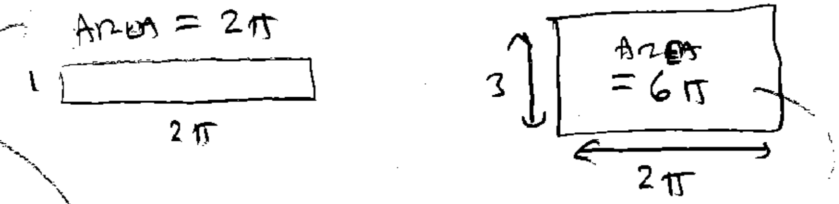


Based on these quick observations, what can you conclude about the shaded area?

SINCE $-1 \le \sin(x) \le 1$

$+2 \qquad +2 \qquad +2$

WE HAVE $\underset{m}{1} \le \sin(x)+2 \le \underset{M}{3}$



THUS

$$\boxed{2\pi \le \int_0^{2\pi} \sin(x)+2\,dx \le 6\pi}$$